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Hyperelastic constitutive modeling under finite strain

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Abstract

A strain energy density function for isotropic higher order elasticity is developed. The strain energy density is decomposed into a compressibility component being a generalization of the Simo and Pister [Comput. Methods Appl. Mech. Eng. 46, 201–215] proposal for neo-Hookean elasticity, and an incompressibility component being the generalized Mooney expression. A general constitutive relationship for the second Piola Kirchhoff and Eulerian stress tensor for higher order elasticity is then derived from the proposed strain energy density. Constitutive relationships for the principal Lagrangian and Eulerian physical stresses in terms of the principal stretches are also developed. Predictions based on the proposed strain energy density are compared with experimental results including incompressible rubber-like materials under homogeneous strain, compressible materials under high hydrostatic compression, and measured volume changes in rubber and foam under large deformation uniaxial tension.

Keywords: Finite strain; Elasticity; Hyperelasticity; Rubber; Elastomers; Foam; Polymers; Biological tissues

1. Introduction

There are several material groups such as elastomers, polymers, foams and biological tissues which can undergo large deformations without permanent set, and hence exhibit large nonlinear elastic behaviour. The nonlinear elastic behaviour under load or prescribed displacement can be modelled using either a physical description of the molecular interplay through theories such as the classical Gaussian theory, slip-link or macromolecular network theories as discussed by Treloar (1975), Boyce and Arruda (2000), Bischoff et al. (2000) and Meissner and Matějka (2002), or by a phenomenological approach. The strain energy expression formulated using a molecular approach is often complex and material specific. In the phenomenological approach, material is treated as a continuum and a strain energy density is postulated, usually in terms of the deformation invariants, generally strain or stretch invariants. Several material parameters are usually needed to reflect the nonlinearity in the load stretch relationships. Typically for

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a rubber-like material under tension, the load stretch response will display an S-shaped behaviour with stiffening at large stretches. The number of material parameters needed will be related to the level of nonlinearity, and whether one loading regime (for example uniaxial tension) or a more general loading state is being modelled.

There are many proposed strain energy density expressions in the literature. These can be grouped into those dealing with incompressible materials and those extended to deal with compressibility. They can further be split depending on the material group being modelled, whether for example the material under consideration is a rubber, polymer, foam or biological tissue. They can also be grouped on whether or not they satisfy the Valanis–Landel hypothesis (see Ogden, 1997; Treloar, 1975). The most widely cited strain energy expressions are the Mooney–Rivlin, Ogden (1997) and Blatz and Ko (1962) models. Descriptions of many of the proposed models can be founded in Treloar (1975), Beatty (1987), Ogden (1997), Lambert-Diani and Rey (1999), El-Lawindy and El-Guiziri (2000), Boyce and Arruda (2000), Bischoff et al. (2000), Bradley et al. (2001), and Meissner and Špirková (2002). A good phenomenological model is one that can give good comparison with experimental results for any stress state with one set of material parameters, gives stable results for all loadings, is applicable to a wide range of materials, and can be used to derive the constitutive relationship for a chosen stress tensor in general coordinates. Attard (2003) proposed a strain energy density for isotropic hyperelastic materials which consisted of the general Mooney (1940) expression for higher order elasticity for the incompressibility component and a generalization of the Simo and Pister (1984) proposal for the compressibility component. The purpose of this paper is to extend the original derivation by detailing the general constitutive relationships for the second Piola Kirchhoff and Eulerian stress tensors in terms of the metric tensor, and to verify the formulation by applying it to a wide variety of materials including rubbers and foams under several loading states.

2. Continuum kinematics

The difference between the square of the length of the differential line element vector in the deformed state $\mathbf{d\hat{s}} = \hat{\mathbf{g}}_i \mathrm{d}s^i$ and the square of the length of the differential line element vector in the undeformed state $\mathbf{ds} = \mathbf{g}_i \mathrm{d}s^i$ is used in as a measure of the state of deformation (Ogden, 1997). That is

$$|\mathbf{d\hat{s}}|^2 - |\mathbf{ds}|^2 = \mathbf{ds} \cdot (\mathbf{C} - \mathbf{I}) \cdot \mathbf{ds} = (\hat{g}_{ik} g^{kj} - \delta_i^j) \mathrm{d}s^i \mathrm{d}s_j \tag{1}$$

where g^{kj} are contravaraiant components of the metric tensor in the undeformed state, $\hat{g}_{ik}$ are covariant components of the metric tensor in the deformed state, δ_j^i is the Kronecker delta, $\mathbf{I} = \mathbf{g}_i \otimes \mathbf{g}^i$ is the identity tensor and $\mathbf{C}$ is the right Cauchy–Green deformation tensor defined by

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}} \mathbf{F} = \hat{g}_{ij} \mathbf{g}^i \otimes \mathbf{g}^j \tag{2}$$

The right Cauchy–Green deformation tensor is symmetric and positive definite. In the above equation, $\mathbf{F} = \hat{\mathbf{g}}_i \otimes \mathbf{g}^i$ is the deformation gradient tensor. Note i, j are indices which take on the values 1, 2 and 3. The convention is adopted where a repeated index is used to imply summation. A bracketed index indicates that the summation convention is suppressed. Using the spectral representation (see Ogden, 1997) of right Cauchy–Green deformation tensor we can write

$$\mathbf{C} = \sum_{i=1}^{3} (\lambda_i)^2 \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)} \quad \mathbf{I} = \mathbf{u}^i \otimes \mathbf{u}^i \tag{3}$$

where $\mathbf{u}^{(i)}$ is the ith normalized eigenvector of $\mathbf{C}$ and λ_i is the ith real positive eigenvalue which also represents the principal relative stretch. There are many tensor invariants which can be written in terms of the

metric tensor in the undeformed and deformed state, as well as the relative stretches. The most common quoted triad of invariants are the principal invariants of the right Cauchy–Green deformation tensor (I_1, I_2, I_3) given by

$$I_1 = \mathrm{tr}(\mathbf{C}) = g^{ij}\hat{g}_{ij} = (\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2 \tag{4}$$

$$I_2 = 1/2\left\{(\mathrm{tr}\,\mathbf{C})^2 - \mathrm{tr}\,\mathbf{C}^2\right\} = g_{ij}\hat{g}^{ij}I_3 = 1/2(I_1^2 - g^{ki}g^{lj}\hat{g}_{kj}\hat{g}_{li}) = (\lambda_1\lambda_2)^2 + (\lambda_1\lambda_3)^2 + (\lambda_2\lambda_3)^2 \tag{5}$$

$$I_3 = J^2 = \frac{\hat{g}}{g} = 1/6\left\{(\mathrm{tr}\,\mathbf{C})^3 - 3\mathrm{tr}\,\mathbf{C}\mathrm{tr}\,\mathbf{C}^2 + 2\mathrm{tr}\,\mathbf{C}^3\right\} = 1/6(I_1^3 - 3I_1 g^{ki}g^{lj}\hat{g}_{kj}\hat{g}_{li} + 2g^{mi}g^{nj}g^{ok}\hat{g}_{jm}\hat{g}_{kn}\hat{g}_{io})$$
$$= (\lambda_1\lambda_2\lambda_3)^2 \tag{6}$$

where λ_1, λ_2 and λ_3 are the principal stretches, $g = \det(g_{ij})$ and $\hat{g} = \det(\hat{g}_{ij})$. The invariant I_1 represents the sum of the squares of relative ratios of the three distinct sides of the deformed parallelepiped, I_2 the sum of the relative ratios of the squares of the three distinct surface areas of the deformed parallelepiped and I_3 the relative ratio of the square of the volume of the deformed parallelepiped. Associated with this set of invariants are the inverse principal invariants given by

$$I_{-1} = \mathrm{tr}(\mathbf{C}^{-1}) = \frac{I_2}{I_3} \quad I_{-2} = \frac{I_1}{I_3} \quad I_{-3} = \frac{1}{I_3} \tag{7}$$

Another set of invariants which are characterized by having no coupling terms in the principal stretches and only involving the principal stretches to even powers are defined by

$$\begin{aligned} L_1 &= \mathrm{tr}(\mathbf{C}) = g^{ij}\hat{g}_{ji} = (\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2 \\ L_2 &= \mathrm{tr}(\mathbf{C}^2) = g^{ki}g^{lj}\hat{g}_{jk}\hat{g}_{il} = (\lambda_1)^4 + (\lambda_2)^4 + (\lambda_3)^4 \\ L_3 &= \mathrm{tr}(\mathbf{C}^3) = g^{mi}g^{nj}g^{ok}\hat{g}_{jm}\hat{g}_{kn}\hat{g}_{io} = (\lambda_1)^6 + (\lambda_2)^6 + (\lambda_3)^6 \end{aligned} \tag{8}$$

$$\begin{aligned} L_{-1} &= \mathrm{tr}(\mathbf{C}^{-1}) = g_{ij}\hat{g}^{ji} = \frac{1}{(\lambda_1)^2} + \frac{1}{(\lambda_2)^2} + \frac{1}{(\lambda_3)^2} \\ L_{-2} &= \mathrm{tr}(\mathbf{C}^{-2}) = g_{ki}g_{lj}\hat{g}^{jk}\hat{g}^{il} = \frac{1}{(\lambda_1)^4} + \frac{1}{(\lambda_2)^4} + \frac{1}{(\lambda_3)^4} \\ L_{-3} &= \mathrm{tr}(\mathbf{C}^{-\mathbf{3}}) = g_{mi}g_{nj}g_{ok}\hat{g}^{jm}\hat{g}^{kn}\hat{g}^{io} = \frac{1}{(\lambda_1)^6} + \frac{1}{(\lambda_2)^6} + \frac{1}{(\lambda_3)^6} \end{aligned} \tag{9}$$

General invariants L_n and L_{-n} of this form with principal stretches to any even power $2n$ are denoted here by

$$L_n = \mathrm{tr}(\mathbf{C}^n) = (\lambda_1)^{2n} + (\lambda_2)^{2n} + (\lambda_3)^{2n} \quad L_{-n} = \mathrm{tr}(\mathbf{C}^{-\mathbf{n}}) = (\lambda_1)^{-2n} + (\lambda_2)^{-2n} + (\lambda_3)^{-2n} \tag{10}$$

with the understanding that n is used as an index in L_n and L_{-n}.

3. Isotropic hyperelastic strain energy density

The variation of the strain energy density δU with respect to the initial volume is related to the variation in work δW by the equation (Renton, 2002):

$$\delta(U\mathrm{d}V) = \delta(W) \tag{11}$$

This equation can be used to establish the constitutive relationship for a hyperelastic material, between a stress tensor and measures of deformation not necessarily the conjugate strain (see Attard, 2003). For the second Piola Kirchhoff stress tensor $\boldsymbol{\pi} = \pi^{ij}\mathbf{g_i} \otimes \mathbf{g_j}$ and the Eulerian stress tensor $\boldsymbol{\tau} = \tau^{ij}\hat{\mathbf{g}}_\mathbf{i} \otimes \hat{\mathbf{g}}_\mathbf{j}$ we have

$$\begin{aligned} \boldsymbol{\pi} &= 2\frac{\partial U}{\partial \mathbf{C}} \quad \boldsymbol{\tau} J = 2\mathbf{F}\frac{\partial U}{\partial \mathbf{C}}\mathbf{F}^{\mathrm{T}} \\ \pi^{ij} &= \tau^{ij}J = \frac{\partial U}{\partial \hat{g}_{ij}} + \frac{\partial U}{\partial \hat{g}_{ji}} = 2\frac{\partial U}{\partial \hat{g}_{ij}} \end{aligned} \tag{12}$$

while for the principal physical Lagrangian $s_{\mathrm{p}}^{(ii)}$ (engineering stresses) and Eulerian stresses $\varsigma_{\mathrm{p}}^{(ii)}$ (true stresses)

$$s_{\mathrm{p}}^{(ii)} = \frac{\partial U}{\partial \lambda_i} \quad \varsigma_{\mathrm{p}}^{(ii)} = \frac{\lambda_{(i)}}{J}\frac{\partial U}{\partial \lambda_i} \tag{13}$$

Note the Cauchy stress tensor is equivalent to the Eulerian stress tensor but is commonly written with basis $\{\mathbf{g_i}\}$ such that $\boldsymbol{\tau} = \tau^{ij}\hat{\mathbf{g}}_\mathbf{i} \otimes \hat{\mathbf{g}}_\mathbf{j} = \sigma^{ij}\mathbf{g_i} \otimes \mathbf{g_j}$ where σ^{ij} are the contravariant Cauchy stress components with respect to undeformed tangent base vectors. Attard (2003) discusses several postulates that the strain energy density must satisfy and these are summarized here. The strain energy density must be non-negative for all deformations and invariant. The strain energy density can be written as a function of either the stretch or strain invariants and because of isotropy be symmetrical in form with respect to the principal stretches λ_1, λ_2 and λ_3. The strain energy density must have a zero minimum value at the undeformed state ($\lambda_1 = 1$, $\lambda_2 = 1$ and $\lambda_3 = 1$). The minimum condition guarantees that the material is stress-free in the undeformed state. Hence

$$\begin{aligned} &\left(\frac{\partial U}{\partial \lambda_i}\right)_{\text{undeformed state}} = 0 \quad \left(\frac{\partial^2 U}{\partial \lambda_i^2}\right)_{\text{undeformed state}} > 0 \\ &\left(\frac{\partial^2 U}{\partial \lambda_i \partial \lambda_j}\right)_{\text{undeformed state}} \geqslant 0 \quad i \neq j \; i,j = 1,2,3 \end{aligned} \tag{14}$$

At a singularity ($\lambda_i = 0$) or ($\lambda_i \to \infty$), the strain energy density must approach positive infinity. Stresses, on the other hand, should approach negative infinity at singularity ($\lambda_i = 0$) and positive infinity for very large deformations ($\lambda_i \to \infty$).

The strain energy density is assumed to be decomposed into two components, one U_{incomp} associated with incompressibility while the other U_{comp} is associated with the compressibility or specific volume change. The compressibility component is associated with the application of a hydrostatic pressure. It is assumed that the incompressibility component should involve no coupling of the principal stretches. This assumption will be discussed further when we look at the constitutive relationship of the principal physical stresses.

It was proposed in Attard (2003) that the incompressibility component could be represented as a linear function of the general invariants L_n and L^n which contain even powers of the principal stretches and the reciprocal of the principal stretches, hence:

$$\begin{aligned} U_{\text{incomp}} &= \sum_{n=1}^{r} \frac{A_n}{2n}\mathrm{tr}(\mathbf{C}^n - \mathbf{I}) + \frac{B_n}{2n}\mathrm{tr}(\mathbf{C}^{-n} - \mathbf{I}) = \sum_{n=1}^{r}\left\{\frac{A_n}{2n}(L_n - 3) + \frac{B_n}{2n}(L_{-n} - 3)\right\} \\ &= \sum_{n=1}^{r}\left\{\frac{A_n}{2n}\left[(\lambda_1)^{2n} + (\lambda_2)^{2n} + (\lambda_3)^{2n} - 3\right] + \frac{B_n}{2n}\left[(\lambda_1)^{-2n} + (\lambda_2)^{-2n} + (\lambda_3)^{-2n} - 3\right]\right\} \end{aligned} \tag{15}$$

where A_n and B_n are material constants, and r is the termination point of the summation. Eq. (15) is the same as Mooney's (1940) general expression for high order elasticity. The simple (two term) Mooney expression for U_{incomp} results if one considers only A_1 and B_1, hence

$$U_{\text{incomp}} = 1/2\{A_1\text{tr}(\mathbf{C}-\mathbf{I}) + B_1\text{tr}(\mathbf{C}^{-1}-\mathbf{I})\} = 1/2\{A_1(L_1-3) + B_1(L_{-1}-3)\} \tag{16}$$

This expression is often rewritten in the literature and referred to as the Mooney–Rivlin expression, by substituting $I_1 = L_1 \quad I_2 = L_{-1}$ for an incompressible material, hence Eq. (16) is rewritten as

$$U_{\text{incomp}} = 1/2\{A_1(I_1-3) + B_1(I_2-3)\}. \tag{17}$$

The expression for U_{incomp} given in Eq. (15) satisfies the Valanis–Landel hypothesis that the strain energy density for incompressible isotropic materials should be capable of representation as the sum of three separate but identical functions of each of the individual principal stretches (see Ogden, 1997; Treloar, 1975). Because of isotropy the compressibility component U_{comp} must be a function of the volumetric dilation through the invariant J. The compressibility component is split into two terms of the form $U_{\text{comp}}(J) = U(J) - \left(\sum_{n=1}^{r}\{A_n - B_n\}\right)\ln J$. The logarithmic term is needed (provided $\left(\sum_{n=1}^{r}\{A_n - B_n\}\right) \neq 0$) so that, the material is stress free at the undeformed state. Simo and Pister (1984) remarked that the compressibility component of the strain energy density should approach infinity at both a singularity $J \to 0$ and infinite volume change $J \to \infty$ and proposed $U_{\text{comp}}(J) = 1/2\Lambda(\ln J)^2 - G\ln J$ for neo-Hookean isotropic elasticity where G is the shear modulus and Λ the Lamé constant. The Simo and Pister (1984) proposal is generalized here. The proposed strain energy density in Attard (2003) for an isotropic hyperelastic material is

$$\begin{aligned} &U = U_{\text{incomp}} + U_{\text{comp}} \\ &U_{\text{incomp}} = \sum_{n=1}^{r}\frac{A_n}{2n}\text{tr}(\mathbf{C}^n - \mathbf{I}) + \frac{B_n}{2n}\text{tr}(\mathbf{C}^{-n} - \mathbf{I}) \quad U_{\text{comp}} = \left\{\sum_{n=1}^{s}\frac{\Lambda_n}{2n}(\ln J)^{2n} - \left(\sum_{n=1}^{r}A_n - B_n\right)\ln J\right\} \end{aligned} \tag{18}$$

where Λ_n are material constants; r and s are termination points of the summation. Another way of splitting the strain energy density would be into non-coupled terms $U_{\text{noncoupled}}$ and an interaction or coupling component U_{coupled} with regard to the principal stretches. The strain energy would then be written as

$$\begin{aligned} &U = U_{\text{noncoupled}} + U_{\text{coupled}} \\ &U_{\text{noncoupled}} = \sum_{n=1}^{r}\frac{A_n}{2n}\text{tr}(\mathbf{C}^n - \mathbf{I}) + \frac{B_n}{2n}\text{tr}(\mathbf{C}^{-n} - \mathbf{I}) - \left(\sum_{n=1}^{r}A_n - B_n\right)\ln J \\ &U_{\text{coupled}} = \sum_{n=1}^{s}\frac{\Lambda_n}{2n}(\ln J)^{2n} \end{aligned} \tag{19}$$

Considering material stability, the material constants must satisfy

$$\begin{aligned} &\left(\frac{\partial^2 U}{\partial\lambda_i^2}\right)_{\text{undeformed state}} = \sum_{n=1}^{r}\{2n(A_n + B_n)\} + \Lambda_1 = E_{\text{t}} > 0 \\ &\left(\frac{\partial^2 U}{\partial\lambda_i\partial\lambda_j}\right)_{\text{undeformed state}} = \Lambda_1 \geqslant 0 \end{aligned} \tag{20}$$

where $E_{\text{t}} = \left(\frac{\partial s_{\text{p}}^{(ii)}}{\partial\lambda_i}\right)_{\text{undeformed state}}$ is the tangent modulus for an isotropic material at the undeformed state. Further, the specified material constants should satisfy the following for physically acceptable stable results:

$$\lambda_i \to 0 \quad U \to \infty \therefore B_r > 0 \quad \text{provided } \mathrm{tr}(\mathbf{C}^{-r} - \mathbf{I}) \neq 0 \quad \forall \lambda_i \text{ else } B_{r-1} > 0 \text{ etc.}$$
$$\text{If } \forall B_n = 0 \quad \text{then } \sum_{n=1}^{r} A_n > 0 \quad \text{and } \Lambda_s \geqslant 0 \tag{21}$$

$$\lambda_i \to \infty \quad U \to \infty \therefore A_r > 0 \quad \text{provided } \mathrm{tr}(\mathbf{C}^{r} - \mathbf{I}) \neq 0 \quad \forall \lambda_i \text{ else } A_{r-1} > 0 \text{ etc.}$$
$$\text{If } \forall A_n = 0 \quad \text{then } \sum_{n=1}^{r} B_n > 0 \quad \text{and } \Lambda_s \geqslant 0 \tag{22}$$

It must be emphasized here, that the stability conditions Eqs. (20)–(22), to be satisfied by the material constants do not guarantee the positive definiteness of the strain energy and this requirement must be checked for each particular set of material parameters chosen.

When a material is Hookean at infinitesimal strain, the material constants are related to the shear modulus G, bulk modulus K and the Lamé constant Λ by

$$G = \frac{E}{2(1+\mu)} = \sum_{n=1}^{r}\Big\{n(A_n + B_n)\Big\}, \quad \Lambda = \frac{E\mu}{(1+\mu)(1-2\mu)} = \Lambda_1, \quad K = \Lambda_1 + \frac{2}{3}\sum_{n=1}^{r} n(A_n + B_n) \tag{23}$$

with E being the elastic modulus and μ the Poisson's ratio. Combining Eqs. (20) and (23) we find that for a Hookean material the following must be satisfied:

$$2G + \Lambda = E_t > 0 \quad \text{and} \quad \Lambda \geqslant 0 \tag{24}$$

implying the well-known condition that the Poisson's ratio must be less than 0.5 and greater than −1.

4. Constitutive relationships—principal physical stresses

First, the general expressions for the principal physical Lagrangian and Eulerian stresses are derived. Combining Eqs. (13) and (18) we have

$$\varsigma_{\mathrm{p}}^{(ii)} = s_{\mathrm{p}}^{(ii)}\frac{\lambda_i}{J} = \frac{1}{J}\sum_{n=1}^{r}\Big\{A_n(\lambda_i^{2n} - 1) - B_n(\lambda_i^{-2n} - 1)\Big\} + p_{\mathrm{v}} = \frac{1}{J}\sum_{n=1}^{r}\Big\{A_n\lambda_i^{2n} - B_n\lambda_i^{-2n}\Big\} + p_{\mathrm{T}} \tag{25}$$

where p_{T} is a hydrostatic pressure component of the principal stresses defined by

$$p_{\mathrm{T}} = \sum_{n=1}^{s}\frac{\Lambda_n}{J}(\ln J)^{2n-1} - \sum_{n=1}^{r}\left\{\frac{A_n - B_n}{J}\right\} = p_{\mathrm{v}} + p_{\mathrm{o}} \tag{26}$$

where $p_{\mathrm{o}} = -\sum_{n=1}^{r}\left\{\frac{A_n - B_n}{J}\right\}$ and $p_{\mathrm{v}} = \sum_{n=1}^{s}\frac{\Lambda_n}{J}(\ln J)^{2n-1}$ is the hydrostatic pressure associated with volumetric dilation. Looking at Eqs. (18) and (25), the principal physical Lagrangian stresses have the form

$$s_{\mathrm{p}}^{(ii)} = \frac{\partial U_{\mathrm{noncoupled}}}{\partial \lambda_i} + \frac{\partial U_{\mathrm{coupled}}}{\partial \lambda_i} = f(A_n, B_n, \lambda_i) + \frac{Jp_{\mathrm{v}}(\Lambda_n, J)}{\lambda_i} \tag{27}$$

It follows that any interaction between the principal physical stresses will therefore only come as a result of a hydrostatic pressure associated with volumetric dilation.

The constitutive law for the principal physical Lagrangian and Eulerian stresses for a Hookean material based on Eq. (25) is therefore

$$\varsigma_{\mathrm{p}}^{(ii)} = s_{\mathrm{p}}^{(ii)}\frac{\lambda_i}{J} = \frac{G}{J}(\lambda_i^2 - 1) + p_{\mathrm{v1}} \tag{28}$$

with p_{v1} being equal to $\frac{\Lambda \ln J}{J}$ (note $A_1 = G$ and $\Lambda_1 = \Lambda$).

When dealing with an exactly incompressible material ($J = 1$), stresses in general can not be determined from the strain energy as one can have a hydrostatic pressure which does not contribute to the strain energy. Vulcanized rubbers for example can be considered as almost incompressible for various loading states. The Poisson's ratio measured by Kawabata et al. (1981) on an isoprene rubber vulcanizate was 0.499914 demonstrating the almost incompressible nature of rubber (excluding loadings involving very large hydrostatic pressures). Stresses can be determined, however, for some simple stress states where it is known that one or more of the principal stresses are zero, say as in the case of uniaxial tension. Let us consider the mean normal principal physical Eulerian stresses derived from Eq. (25) that is

$$p = 1/3(\varsigma_{\mathrm{p}}^{11} + \varsigma_{\mathrm{p}}^{22} + \varsigma_{\mathrm{p}}^{33}) = 1/3\frac{1}{J}\sum_{n=1}^{r}\left\{A_n \mathrm{tr}(\mathbf{C}^n - \mathbf{I}) - B_n \mathrm{tr}(\mathbf{C}^{-\mathbf{n}} - \mathbf{I})\right\} + p_{\mathrm{v}} \tag{29}$$

Now let us rewrite the equation for the principal physical stresses using Eq. (29) in the form

$$\varsigma_{\mathrm{p}}^{(ii)} = s_{\mathrm{p}}^{(ii)}\frac{\lambda_i}{J} = \frac{1}{J}\sum_{n=1}^{r}\left\{A_n(\lambda_i^{2n} - 1/3\mathrm{tr}(\mathbf{C}^n)) - B_n(\lambda_i^{-2n} - 1/3\mathrm{tr}(\mathbf{C}^{-n}))\right\} + p \tag{30}$$

We can eliminate the mean normal stress by writing

$$\begin{aligned}
\varsigma_{\mathrm{p}}^{11} - \varsigma_{\mathrm{p}}^{22} &= \frac{1}{J}(s_{\mathrm{p}}^{11}\lambda_1 - s_{\mathrm{p}}^{22}\lambda_2) = \frac{1}{J}\sum_{n=1}^{r}\left\{A_n(\lambda_1^{2n} - \lambda_2^{2n}) - B_n(\lambda_1^{-2n} - \lambda_2^{-2n})\right\} \\
\varsigma_{\mathrm{p}}^{22} - \varsigma_{\mathrm{p}}^{33} &= \frac{1}{J}(s_{\mathrm{p}}^{22}\lambda_2 - s_{\mathrm{p}}^{33}\lambda_3) = \frac{1}{J}\sum_{n=1}^{r}\left\{A_n(\lambda_2^{2n} - \lambda_3^{2n}) - B_n(\lambda_2^{-2n} - \lambda_3^{-2n})\right\} \\
\varsigma_{\mathrm{p}}^{33} - \varsigma_{\mathrm{p}}^{11} &= \frac{1}{J}(s_{\mathrm{p}}^{33}\lambda_3 - s_{\mathrm{p}}^{11}\lambda_1) = \frac{1}{J}\sum_{n=1}^{r}\left\{A_n(\lambda_3^{2n} - \lambda_1^{2n}) - B_n(\lambda_3^{-2n} - \lambda_1^{-2n})\right\}
\end{aligned} \tag{31}$$

For the special case of bi-axial loading of an incompressible material producing $\varsigma_{\mathrm{p}}^{33} = 0$ and $\lambda_3 = \frac{1}{\lambda_1\lambda_2}$ we can further simplify Eq. (31) to

$$\varsigma_{\mathrm{p}}^{11} - \varsigma_{\mathrm{p}}^{22} = s_{\mathrm{p}}^{11}\lambda_1 - s_{\mathrm{p}}^{22}\lambda_2 = \sum_{n=1}^{r}\left\{A_n(\lambda_1^{2n} - \lambda_2^{2n}) - B_n(\lambda_1^{-2n} - \lambda_2^{-2n})\right\} \tag{32}$$

$$\varsigma_{\mathrm{p}}^{11} = s_{\mathrm{p}}^{11}\lambda_1 = \sum_{n=1}^{r}\left\{A_n(\lambda_1^{2n} - (\lambda_1\lambda_2)^{-2n}) - B_n(\lambda_1^{-2n} - (\lambda_1\lambda_2)^{2n})\right\} \tag{33}$$

$$\varsigma_{\mathrm{p}}^{22} = s_{\mathrm{p}}^{22}\lambda_2 = \sum_{n=1}^{r}\left\{A_n(\lambda_2^{2n} - (\lambda_1\lambda_2)^{-2n}) - B_n(\lambda_2^{-2n} - (\lambda_1\lambda_2)^{2n})\right\} \tag{34}$$

5. Constitutive relationships—stress tensors

The Eulerian and the second Piola Kirchhoff stress tensor can be determined from Eqs. (12) and (18). To derive the partial derivatives of the invariants with respect to the right Cauchy–Green deformation tensor we note the following:

$$\frac{\partial \mathrm{tr}(\mathbf{C}^n)}{\partial \mathbf{C}} = n\mathbf{C}^{n-1} \tag{35}$$

To prove this formula, we invoke the spectral representation of $\mathbf{C}^n = \sum_{i=1}^{3} (\lambda_i^2)^n \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}$ and hence we can write

$$\frac{\partial \mathrm{tr}(\mathbf{C}^n)}{\partial \mathbf{C}} = \sum_{i=1}^{3} \left\{ \frac{\partial \sum_{m=1}^{3} (\lambda_m^2)^n}{\partial \lambda_i^2} \mathbf{u}^{(\mathbf{i})} \otimes \mathbf{u}^{(\mathbf{i})} \right\} = n\mathbf{C}^{n-1} \tag{36}$$

Substituting Eq. (18) into Eq. (12), using Eq. (35) and noting that $\frac{\partial \ln J}{\partial \mathbf{C}} = 1/2\mathbf{C}^{-1}$, we can write

$$\begin{aligned} \boldsymbol{\pi} &= \sum_{n=1}^{r} (A_n[\mathbf{C}^n - \mathbf{I}] - B_n[\mathbf{C}^{-n} - \mathbf{I}])\mathbf{C}^{-1} + Jp_{\mathrm{v}}\mathbf{C}^{-1} \\ \boldsymbol{\tau} &= \frac{1}{J} \sum_{n=1}^{r} (A_n[\mathbf{B}^n - \mathbf{I}] - B_n[\mathbf{B}^{-n} - \mathbf{I}]) + p_{\mathrm{v}}\mathbf{I} \end{aligned} \tag{37}$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ is the Finger or left Cauchy–Green deformation tensor. The volumetric part of the Eulerian stress tensor is then given by

$$(\boldsymbol{\tau})^{\mathrm{vol}} = 1/3\mathrm{tr}(\tau)\mathbf{I} = \frac{1}{J} \left\{ 1/3 \sum_{n=1}^{r} A_n \mathrm{tr}(\mathbf{B}^n - \mathbf{I}) - B_n \mathrm{tr}(\mathbf{B}^{-n} - \mathbf{I}) \right\} \mathbf{I} + p_{\mathrm{v}}\mathbf{I} \tag{38}$$

being invariant of the coordinate configuration. The deviatoric part of the Eulerian stress tensor is therefore

$$(\boldsymbol{\tau})^{\mathrm{dev}} = \sum_{n=1}^{r} (A_n \mathbf{B}^n - B_n \mathbf{B}^{-n}) - \left\{ 1/3 \sum_{n=1}^{r} A_n \mathrm{tr}(\mathbf{B}^n) - B_n \mathrm{tr}(\mathbf{B}^{-n}) \right\} \mathbf{I} \tag{39}$$

One can see that the deviatoric part of the Eulerian stress tensor is not a function of the compressibility material parameters Λ_n or the hydrostatic pressure. The physical Lagrangian s^{ij} (the physical counterpart to $\pi^{ij}\hat{\mathbf{g}}_{\mathbf{j}} \otimes \mathbf{g}_{\mathbf{i}}$) and the physical Eulerian ς^{ij} stresses (the physical counterpart to $\tau^{ij}\hat{\mathbf{g}}_{\mathbf{j}} \otimes \hat{\mathbf{g}}_{\mathbf{i}}$) can now be derived using Eq. (37) from (see Attard, 2003)

$$s^{ij} = J\varsigma^{ij} \frac{\sqrt{\hat{g}^{(ii)}}}{\sqrt{g^{(ii)}}} = J \frac{\sqrt{\hat{g}_{(jj)}}}{\sqrt{g^{(ii)}}} \tau^{ij} \tag{40}$$

For a neo-Hookean material Eq. (37) reduces to

$$\begin{aligned} \boldsymbol{\pi} &= G[\mathbf{I} - \mathbf{C}^{-1}] + \Lambda \ln J \mathbf{C}^{-1} \quad \boldsymbol{\tau} J = G[\mathbf{B} - \mathbf{I}] + \Lambda \ln J \mathbf{I} \\ \pi^{ij} &= \tau^{ij} J = G[g^{ij} - \hat{g}^{ij}] + \Lambda \ln J \hat{g}^{ij} \end{aligned} \tag{41}$$

6. Comparison with experiments

The experimental results presented in the following figures have generally been obtained by digitizing from scanned graphs taken from the literature unless tabulated results were available. The number of material parameters needed to describe the test results depends on the level of non-linearity of the load versus stretch relationships and whether one wants to just model one loading regime say uniaxial tension or if one wants to be able to model several stress states. The material parameters were in general determined by using a least squares regression analysis of the experimental data. This was easily implemented using the commercial package MATLAB. If the shear modulus and/or the bulk modulus are known then Eq. (23)

was also incorporated into the least squares analysis. The constraints defined by Eqs. (20)–(22) are also checked.

7. Incompressible rubber-like materials

Experimental verification of many of the proposed strain energy density expressions for incompressible materials is usually based on experiments involving a state of pure homogeneous strain. The experiments usually include uniaxial tension, uniaxial compression, equi-biaxial tension and pure shear. Appendix A details the expressions for the physical principal stresses calculated for these various loading states. The datum set of experiments consisting of uniaxial tension, pure shear and equi-biaxial tension are those of Treloar (1944). In Attard (2003) it was shown that a four parameter (A_1, A_2, A_3, B_1) strain energy density based on Eq. (18) gave a good comparison with the experimental results of Treloar (1944). However, if one examines Eq. (A.4) for ς_p^{22} under pure shear, it is evident that more terms are required if one wants to describe any stress state. Ogden (1997) obtained an excellent fit with a six parameter model. A six parameter model will generally be used here for modeling rubber or rubber-like materials unless otherwise stated.

Figs. 1–3 show a comparison with the experimental results on rubber of Treloar (1944), Heuillet and Dugautier (1997) and Kawabata et al. (1981), respectively. All comparisons give a reasonable fit to the experimental results. The material parameters used in the proposed model are given in each of the figures. Fig. 4 shows a further comparison for a silicon rubber under uniaxial compression and biaxial compression taken from Arruda and Boyce (1993). The example of a neoprene film under equi-biaxial tension and uniaxial tension tested by Alexander (1968) is shown in Fig. 5. The neoprene film displays a high level of stiffening with increasing deformation.

Kawabata et al. (1981) also tested their isoprene rubber vulcanizate under biaxial tension for various fixed values of λ_1 while varying λ_2. The material parameters quoted in Fig. 3 were substituted into Eqs. (33) and (34) to estimate the biaxial stress state. Fig. 6 shows a comparison of the predictions for the stress state and those obtained from experiments. Generally the results match very well. The only discrepancy is for the

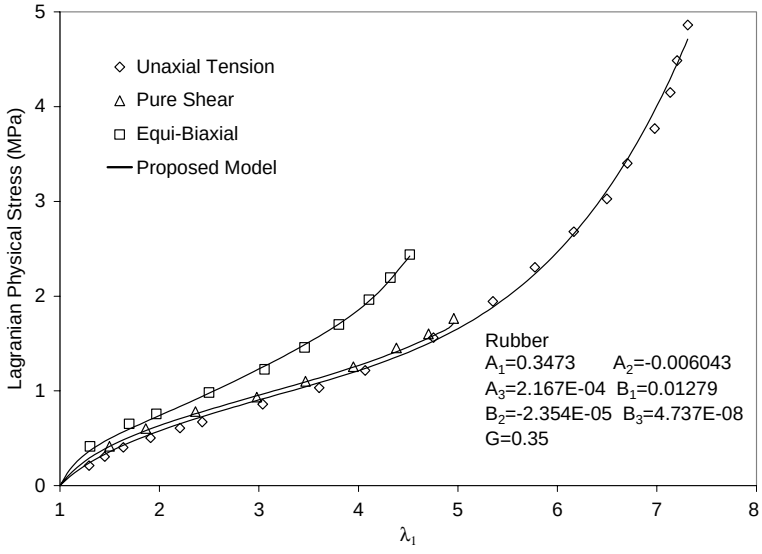

Fig. 1. Comparison with the experimental results of Treloar (1944).

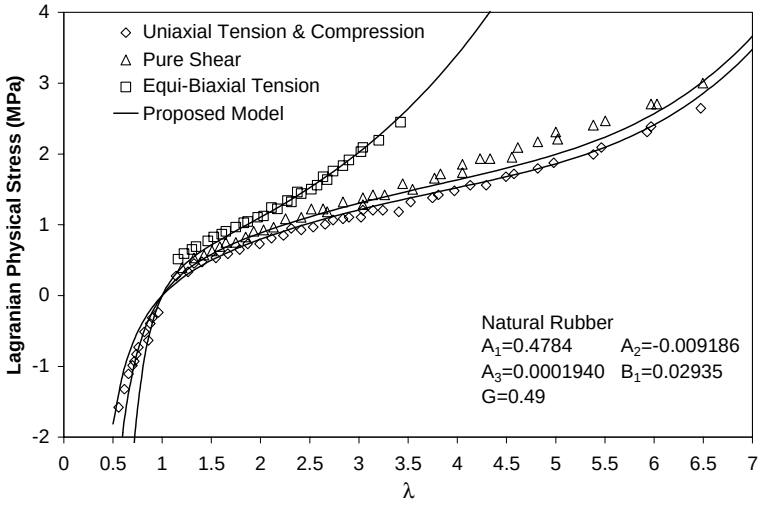

Fig. 2. Comparison with the experimental results of Heuillet and Dugautier (1997).

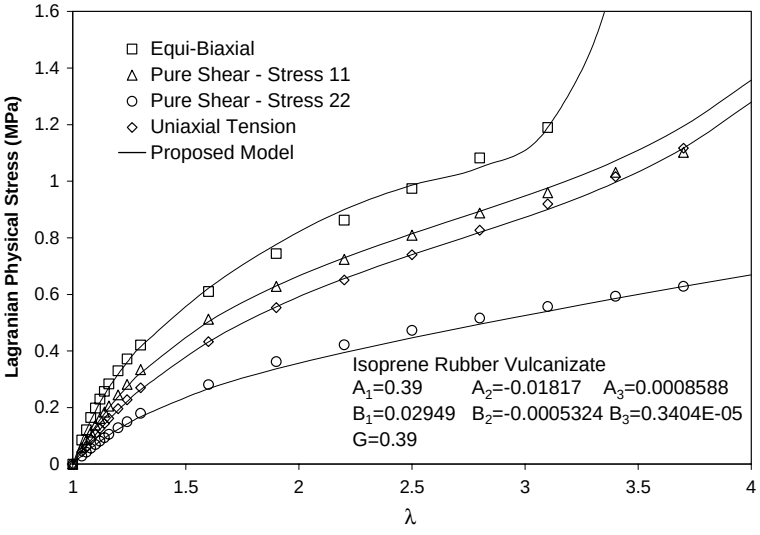

Fig. 3. Comparison with the experimental results of Kawabata et al. (1981).

largest held stretches $\lambda_1 = 3.4$ and 3.7. As will be discussed later this could indicate that the proposed strain energy density does not hold for biaxial loading with large constrained stretch or another explanation could be that the test results could be affected by relaxation in the rubber when held at very large stretch. We will come back to this explanation later.

The Valanis–Landel hypothesis mentioned earlier, states that the strain energy density could be written as three separate independent functions of the principal stretches. Ogden (1979) refers to this as the "separability" of the strain energy density. One of the conclusions that can be surmised from the Valanis–Landel hypothesis is that given a biaxial loading producing stresses say ς_p^{11} and ς_p^{22}, at constant λ_2, the

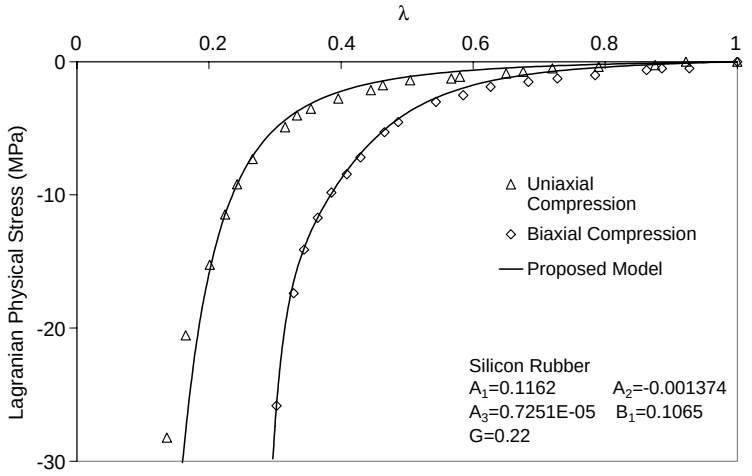

Fig. 4. Comparison with the experimental results of Arruda and Boyce (1993).

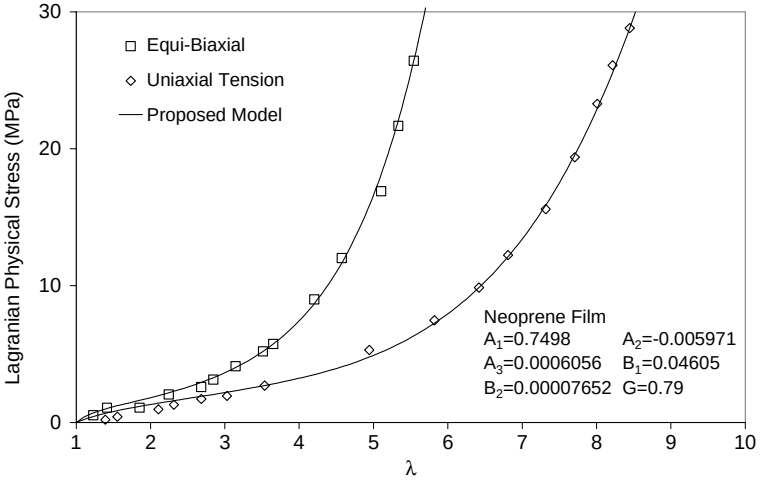

Fig. 5. Comparison with the experimental results of Alexander (1968).

curves as a function of λ_1 representing the difference $\varsigma_{\mathrm{p}}^{11} - \varsigma_{\mathrm{p}}^{22}$, should all have the same shape. All the curves should be able to be superimposed on the curve for pure shear ($\lambda_2 = 1$). To see this consider Eq. (32) rewritten here:

$$\varsigma_{\mathrm{p}}^{11} - \varsigma_{\mathrm{p}}^{22} = s_{\mathrm{p}}^{11}\lambda_1 - s_{\mathrm{p}}^{22}\lambda_2 = \sum_{n=1}^{r} A_n(\lambda_1^{2n} - \lambda_2^{2n}) - B_n(\lambda_1^{-2n} - \lambda_2^{-2n}) \tag{42}$$

It is easy to see that the family of curves for $\varsigma_{\mathrm{p}}^{11} - \varsigma_{\mathrm{p}}^{22}$ should all have the same shape but are shifted vertically for different constant λ_2. Vangerko and Treloar (1978) were able to test this hypothesis experimentally by applying large biaxial stretches up to the order of 5 on two batches of rubber, one containing 3% Sulphur and the other 5%. Comparisons of the measured true stresses versus stretch with the results of

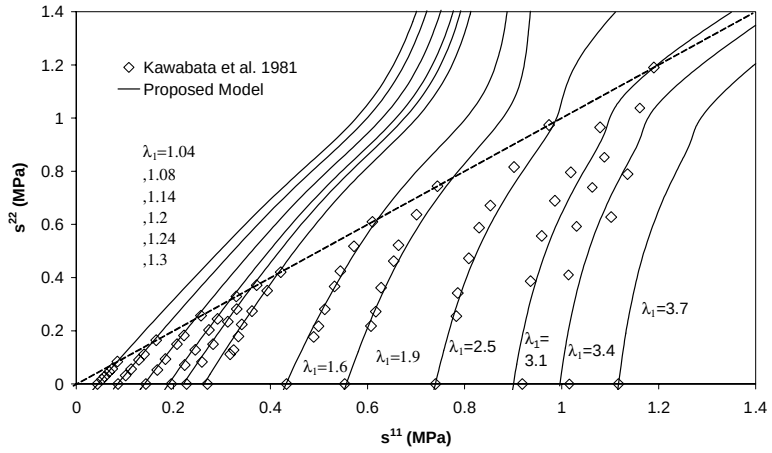

Fig. 6. Comparison with the experimental results of Kawabata et al. (1981). The broken line represents equi-biaxial loading while the horizontal axis represents uniaxial tension.

the proposed model are shown in Figs. 7 and 8. Fig. 9 shows plots of the true stress difference versus stretch λ_1. The prediction of the stress $\varsigma_p^{11} = s_p^{11}\lambda_1$ in the direction of the varying stretch λ_1, compares very well with the test results as shown in Fig. 7. The prediction of the stress $\varsigma_p^{22} = s_p^{22}\lambda_2$ in the direction of the constrained stretch λ_2, generally compares well with the test results, except for the largest constrained stretch $\lambda_2 = 3.36$ for the 3% sulphur rubber and $\lambda_2 = 3.38$ for the 5% sulphur rubber. For these two cases the test results are lower than the predicted values (Fig. 8). Vangerko and Treloar (1978) compared their test results to the prediction of Ogden's (1972) model which also satisfies the Valanis–Landel hypothesis. The predictions of $\varsigma_p^{22} = s_p^{22}\lambda_2$ by the Ogden model were also larger than the test results for these two cases. Vangerko and Treloar (1978) remarked that it is natural to attribute the lower values of $\varsigma_p^{22} = s_p^{22}\lambda_2$ to a "stress-relaxation effect, associated with the fact that the circumferential strain λ_2 in the specimen is maintained during the whole course of an experiment, whereas the axial strain λ_1 attains its maximum value only for a comparatively small fraction of the total time". Fig. 9 shows a comparison of the $\varsigma_p^{11} - \varsigma_p^{22} = s_p^{11}\lambda_1 - s_p^{22}\lambda_2$ curves. If the Valanis–Landel hypothesis holds then the curves should all be parallel. The test results compare well with the predicted except for the two cases of the largest constrained stretch $\lambda_2 = 3.36$ for the 3% sulphur rubber and $\lambda_2 = 3.38$ for the 5% sulphur rubber. Vangerko and Treloar (1978) observed that the value of λ_1 for the equi-biaxial stress state $\varsigma_p^{11} - \varsigma_p^{22} = 0$, should be equal to the maintained stretch λ_2. For the cases $\lambda_2 = 3.36$ for the 3% sulphur rubber and $\lambda_2 = 3.38$ for the 5% sulphur rubber, the test results for λ_1 for which $\varsigma_p^{11} - \varsigma_p^{22} = 0$, were not identical to the maintained stretch. This supports the contention that stress-relaxation has effected the results for the largest maintained stretches and that the Valanis–Landel hypothesis is supported by the experimental results.

8. Compressible materials

8.1. High hydrostatic pressure

The strain energy density model presented in Eq. (18) has been compared to experiments carried out at very high hydrostatic pressures in Attard (2003) to investigate the appropriateness of the compressibility component of the proposed strain energy density. The results presented in Attard (2003) will be summa-

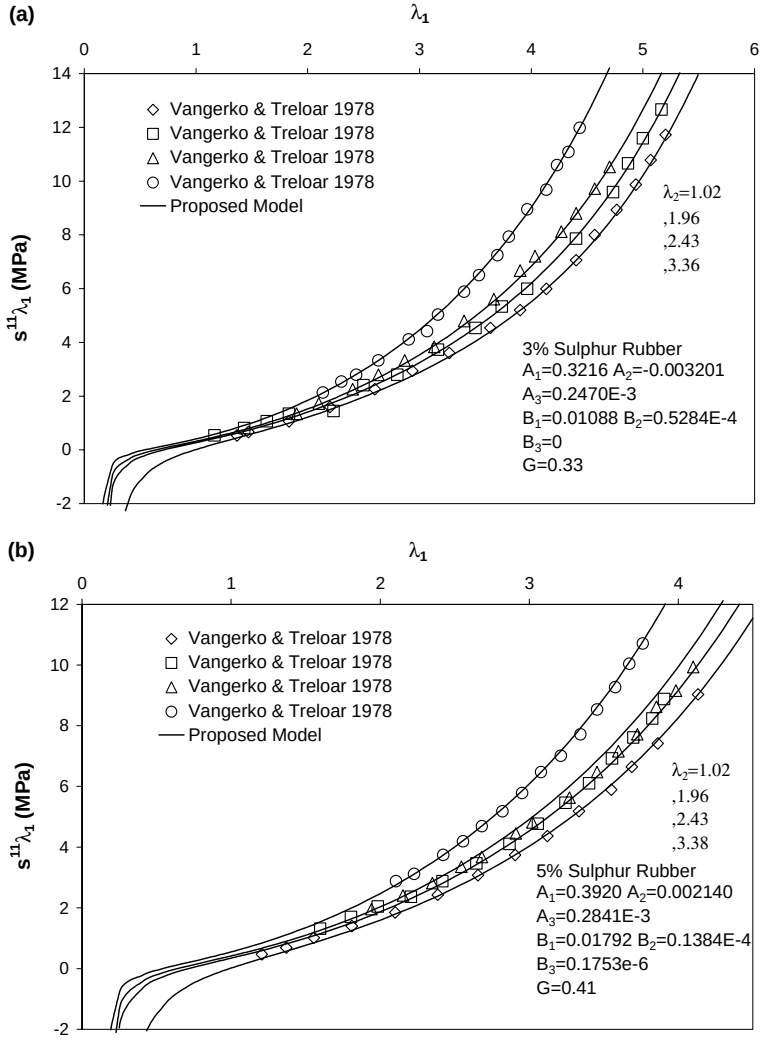

Fig. 7. Comparison with the experimental results of Vangerko and Treloar (1978): (a) 3% sulphur rubber and (b) 5% sulphur rubber.

rized here. In hydrostatic compression tests performed by Adams and Gibson (1930) and Bridgman (1933, 1935, 1945) results were achieved down to values of J of the order of 0.8 (see Ogden, 1997). A three parameter ($A_1 = G$, $\Lambda_1 = \Lambda$, Λ_2) strain energy density expression for compressible materials was considered in Attard (2003) and detailed in the following equation:

$$U = 1/2G(I_1 - 3) + 1/2\Lambda(\ln J)^2 + 1/4\Lambda_2(\ln J)^4 - G\ln J \tag{43}$$

Assuming that under a hydrostatic pressure we make take $\lambda_1 = \lambda_2 = \lambda_3 = J^{1/3}$, the general form of the hydrostatic pressure p_h can then be derived from Eq. (29) as:

$$Jp_h = \sum_{n=1}^{r}\left(A_n J^{2n/3} - B_n J^{-2n/3}\right) + \sum_{n=1}^{s}\Lambda_n(\ln J)^{2n-1} \tag{44}$$

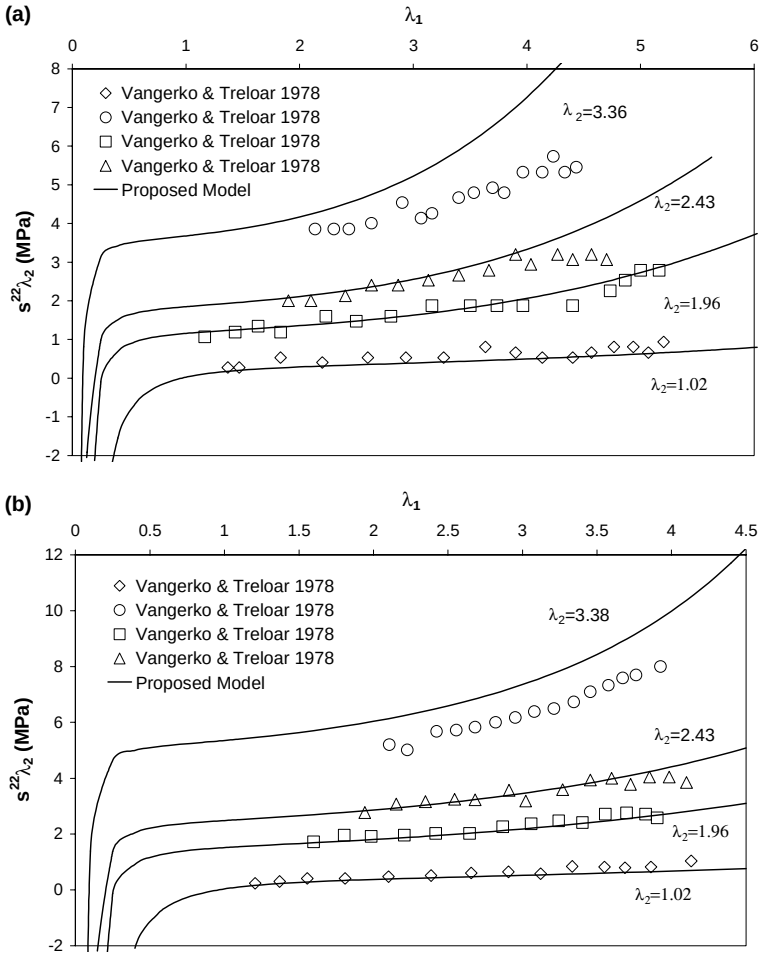

Fig. 8. Comparison with the experimental results of Vangerko and Treloar (1978): (a) 3% sulphur rubber and (b) 5% sulphur rubber.

For the three parameter model considered here we have

$$Jp_{\mathrm{h}} = G(J^{2/3} - 1) + A(\ln J) + A_2(\ln J)^3 \tag{45}$$

Based on the experimental results for the compressibility of sodium and *N*-amyl iodide of Bridgman (1935, 1933), Rubber "A" of Adams and Gibson (1930) and Goodrich D-402 and Koroseal of Bridgman (1945), material parameters were determined and are listed in Table 1. Fig. 10 shows a good comparison between the model predictions with the experimental results.

8.2. Uniaxial tension—volume change in rubber

Attard (2003) investigated the use of the proposed strain energy density to predict volume change under uniaxial tension. The experimental work of Penn (1970) was studied. This example will be reexamined here. Penn (1970) measured the volume change of vulcanized natural gum rubber under uniaxial tension. The stress versus stretch data showed the opposite curvature to that demonstrated by the volume change versus

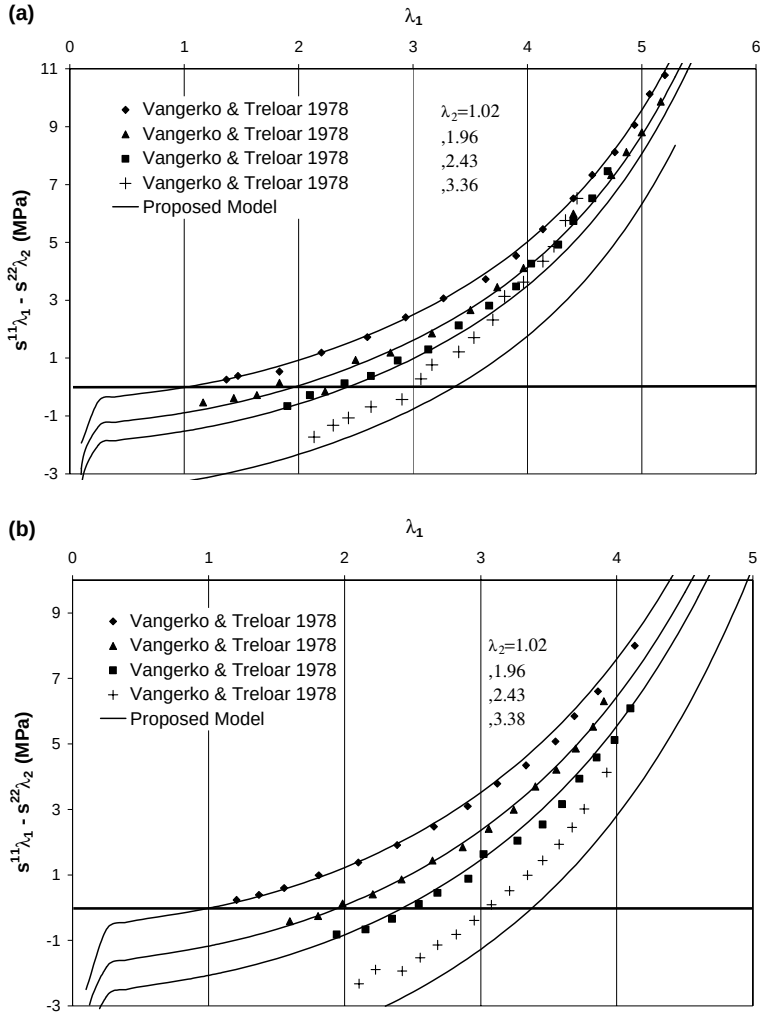

Fig. 9. Comparison with the experimental results of Vangerko and Treloar (1978): (a) 3% sulphur rubber and (b) 5% sulphur rubber.

Table 1
Material parameters

Material parameter	$A_1 = G$	K (GPa)	$\Lambda_1 = \Lambda$ (GPa)	A_2/A_1
Sodium	3.333 GPa	6.3	4.078	11.12
N-amyl iodide	0	1.73	1.73	15.87
Rubber "A"	0	5.2	5.2	44.43
Goodrich D-402	0	3.55	3.55	48.02
Koroseal	0	2.63	2.63	48.02

stretch data (refer to Figs. 11 and 12). At stretches above 1.5 the "two curves deviate significantly when the volume change becomes concave downward while the stress curves upward". Penn (1970) argued that because of this deviation the strain energy density could not be decomposed into the sum of an incompressible and compressible component.

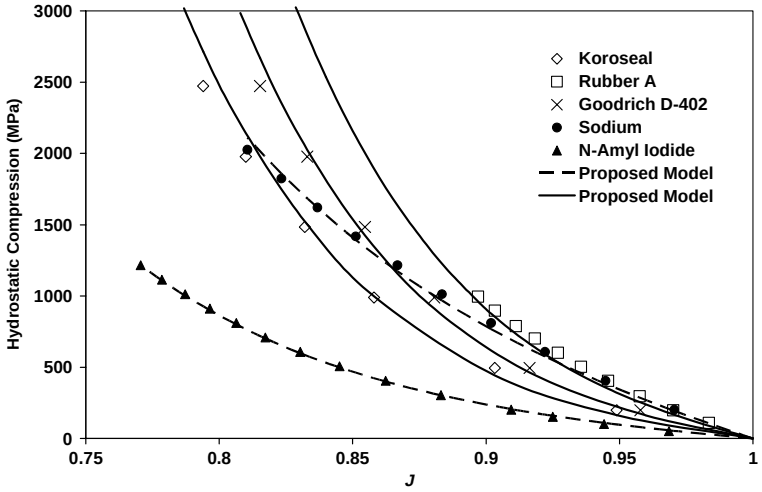

Fig. 10. Comparison with the experimental results of Bridgman (1933, 1935, 1945) and Adams and Gibson (1930).

Attard (2003) used a four parameter strain energy density for this example given by

$$U = 1/2A_1\mathrm{tr}(\mathbf{C}-\mathbf{I}) + 1/2B_1\mathrm{tr}(\mathbf{C}^{-1}-\mathbf{I}) + 1/4A_2\mathrm{tr}(\mathbf{C}^2-\mathbf{I}) + 1/2\Lambda_1(\ln J)^2 - (A_1+A_2-B_1)\ln J \quad (46)$$

The Lagrangian physical principal normal and lateral stresses ($s_{\mathrm{p}}^{22} = s_{\mathrm{p}}^{33} = 0$) based on the four parameter model are therefore

$$s_{\mathrm{p}}^{11}\lambda_1 = A_1(\lambda_i^2-1) + A_2(\lambda_i^4-1) - B_1(\lambda_i^{-2}-1) + \Lambda_1\ln J \quad (47)$$

$$0 = s_{\mathrm{p}}^{22}\lambda_2 = A_1(\lambda_2^2-1) + A_2(\lambda_2^4-1) - B_1(\lambda_2^{-2}-1) + \Lambda_1\ln J \quad (48)$$

Ehlers and Eipper (1998) who examined numerically the lateral strain under uniaxial loading for rubber and rubber-like materials, they observed that certain formulations gave unphysical results. Plots of numerically obtained longitudinal strain versus lateral strain showed unrealistic results as $J \to 0$. The lateral strain first increased as the section was compressed but as the longitudinal strain approached -1 ($J \to 0$), some formulations predicted that the lateral strain would then contract and approach negative infinity. It can be seen from Eq. (48) that as $J \to 0$, the lateral stretch behaviour predicted by the proposed formulation behaves as one would expect with $\lambda_2 \to +\infty$ (note Λ_1, B_1 and A_2 are assumed to be positive). Eq. (48) also reveals that as $\lambda_1 \to +\infty$ the lateral stretch asymptotes towards zero $\lambda_2 \to +0$ and the volume ratio approaches positive infinity $J \to +\infty$.

The material parameter Λ_1 was set to the bulk modulus estimated from Penn (1970) at 2000 MPa. Penn (1970) quoted the Mooney constants as approximately 0.361 and 0.165. These correspond to parameters A_1 and B_1, respectively. The A_1 parameter was set at 0.361 and the remaining two parameters (A_2 and B_1) were estimated using the experimental data. Once the parameters were set, Eq. (48) was numerically solved for the lateral stretch λ_2 for a given value of the longitudinal stretch λ_1. The longitudinal stress defined in Eq. (47) was then calculated for the given longitudinal stretch λ_1 and estimated lateral stretch λ_2. The predictions of the model are compared to the experimental data of Penn (1970) in Figs. 11–13 and show excellent agreement. Fig. 13 demonstrates no unrealistic results are obtained as $\lambda_1 \to \infty$.

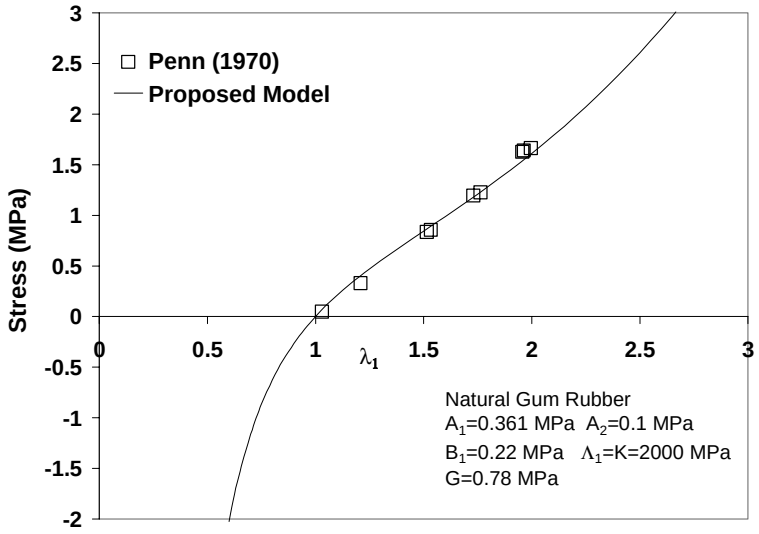

Fig. 11. Comparison with experimental data of Penn (1970).

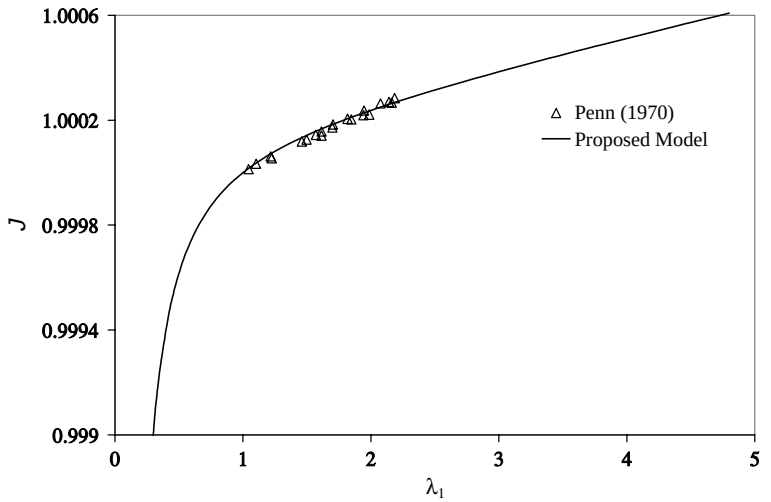

Fig. 12. Comparison of longitudinal stretch versus volume change.

8.3. Uniaxial tension—volume change in polyurethane foam

In the previous example dealing with a natural gum rubber, there was a continuous volume increase with increasing tension. This is generally the case for most compressible materials and the proposed hyperelastic model is based on this premise. The exception seems to be for cellular solids such as polyurethane or open-cell elastomeric foams. Such foams have wide practical applications including cushioning, energy absorption and as noise barriers. Continuing volume dilation has been observed for some polyurethane foams under tension (see Blatz and Ko, 1962). However, uniaxial tension tests carried out by El-Ratal and

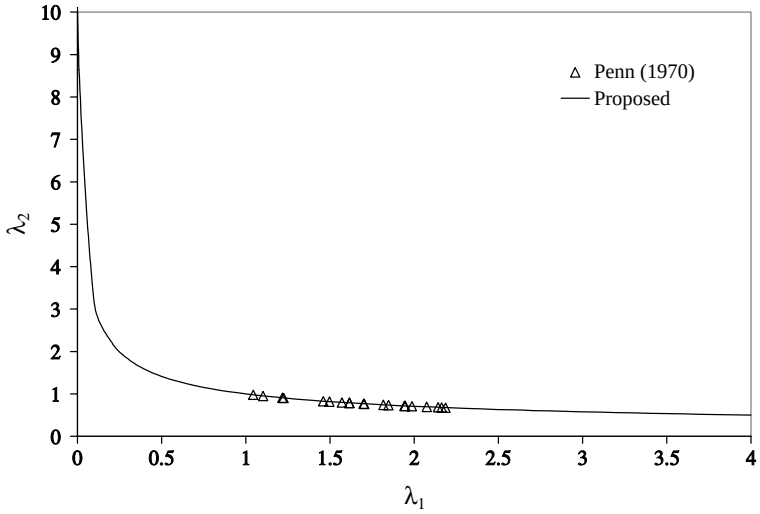

Fig. 13. Longitudinal stretch versus lateral stretch.

Mallick (1996) on a commercial and seat type foam showed an initial increase in volume under tension, but once a maximum was obtained there was volume decrease. The loading during the tests was discontinued when signs appeared of possible tearing. The longitudinal stress versus stretch relationship was highly nonlinear as can be seen in Fig. 14 and has a very different behaviour to that of the natural gum rubber studied by Penn (1970). The proposed hyperelastic model was applied to the uniaxial tension test results of El-Ratal and Mallick (1996). Others to model the test results of El-Ratal and Mallick (1996) have been Jemiolo and Turteltaub (2000) and Murphy and Rogerson (2002). Jemiolo and Turteltaub (2000) presented a parametric study based on a modified elastic strain energy density derived from Ogden's (1997) model.

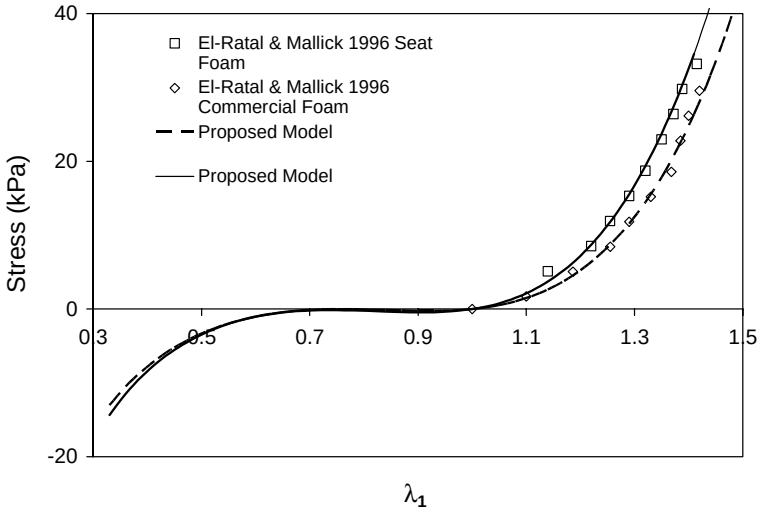

Fig. 14. Comparison of uniaxial tension stress versus longitudinal stretch.

Murphy and Rogerson (2002) extended Blatz and Ko (1962) hypothesis of a power law relationship between the longitudinal stretch and the lateral stretch for the case of uniaxial tension.

To model the two types of foam, a four parameter model (A_1, A_2, A_3 and Λ_1) was employed here. The equations for the Lagrangian physical principal normal stresses are:

$$s_{\mathrm{p}}^{11}\lambda_1 = A_1(\lambda_i^2 - 1) + A_2(\lambda_i^4 - 1) + A_3(\lambda_i^6 - 1) + \Lambda_1 \ln J \tag{49}$$

$$0 = s_{\mathrm{p}}^{22}\lambda_2 = A_1(\lambda_2^2 - 1) + A_2(\lambda_2^4 - 1) + A_3(\lambda_2^6 - 1) + \Lambda_1 \ln J \tag{50}$$

A least squares fit was used to select the material parameters which are given in Table 2. The same numerical approach as in the previous example was employed to solve for the longitudinal stress, and the longitudinal and lateral stretches. Fig. 14 shows a comparison of the experimental stress versus stretch results with those predicted from the hyperelastic model. The comparison is reasonable. Fig. 15 compares the lateral stretch versus longitudinal stretch results. Again one could ascertain that the comparison is reasonable although one would need compression data to further verify the model.

The volume change predicted by the proposed hyperelastic model as shown in Fig. 16, shows a trend which is not evident in the experimental results for stretches close to those beyond the maximum measured in the experiments. In the test results, the volume change shows an increase and then a decrease from a maximum. The proposed hyperelastic model also follows this trend but then shows a steady increase from a minimum. The proposed hyperelastic model with the parameters chosen to model this experiment predicted

Table 2
Material parameters

Material parameter	Seat foam (kPa)	Commercial foam (kPa)
A_1	26.26	25.03
A_2	−44.97	−39.24
A_3	22.72	18.77
Λ_1	3.504	2.268

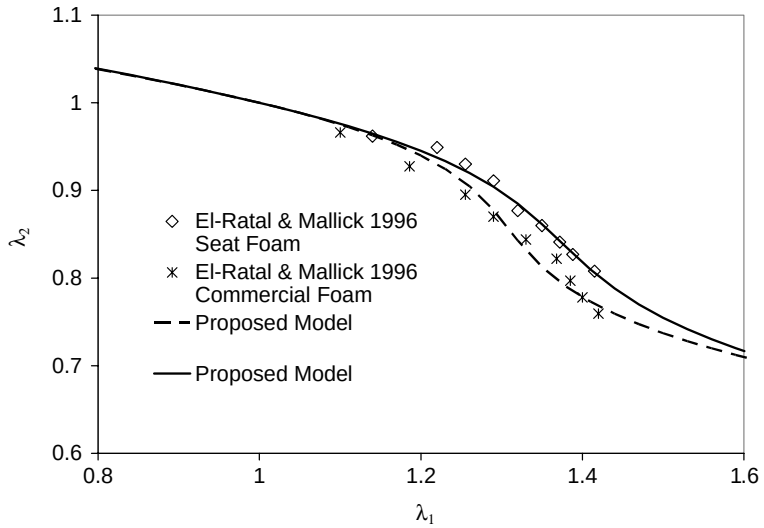

Fig. 15. Comparison of lateral stretch versus longitudinal stretch.

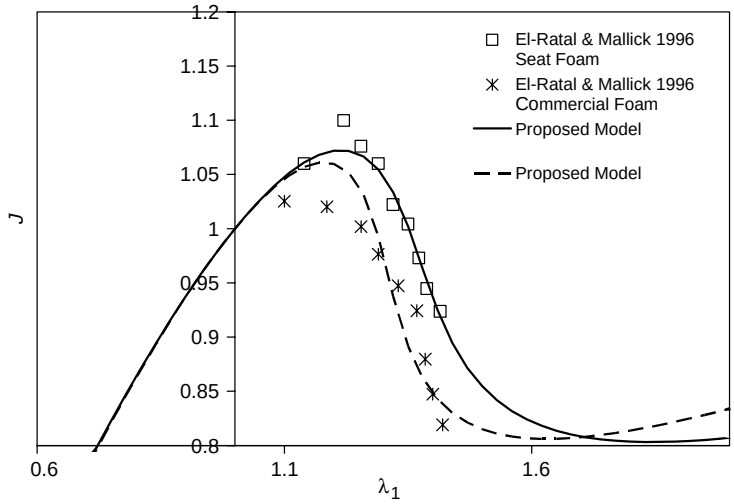

Fig. 16. Comparison of volume change with longitudinal stretch.

that as $\lambda_1 \to +\infty$ the lateral stretch asymptotes towards zero $\lambda_2 \to +0$ and the volume ratio approaches positive infinity $J \to +\infty$. An opposite trend is observed if one looks at the Poisson's ratio defined by $\mu(\lambda_1) = \frac{\lambda_2 - 1}{1 - \lambda_1}$. Fig. 17 shows a comparison of the Poisson's ratio. The test results show an increasing Poisson's ratio simulated correctly by the hyperelastic model. The proposed model, however, shows a maximum Poisson's ratio and then a continuing decrease. Blatz and Ko (1962) suggested that for a highly nonlinear material it is more appropriate to define an apparent Poisson's ratio given by

$$\mu = \frac{-\ln(\lambda_2)}{\ln(\lambda_1)} \tag{51}$$

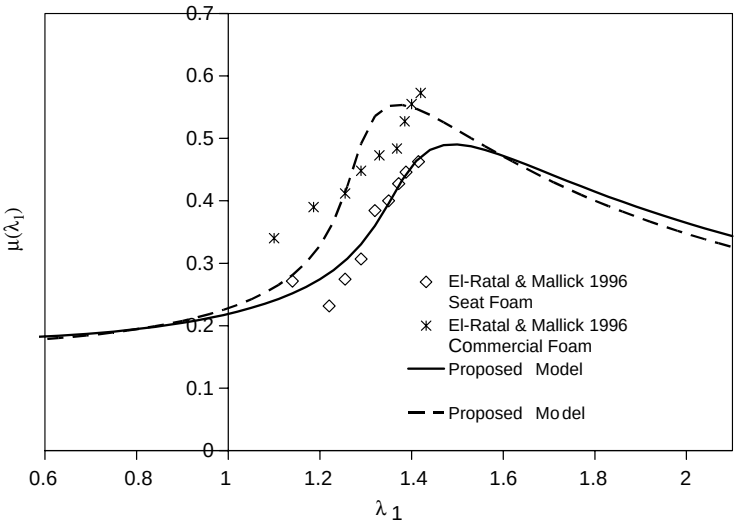

Fig. 17. Comparison of Poisson's ratio versus longitudinal stretch.

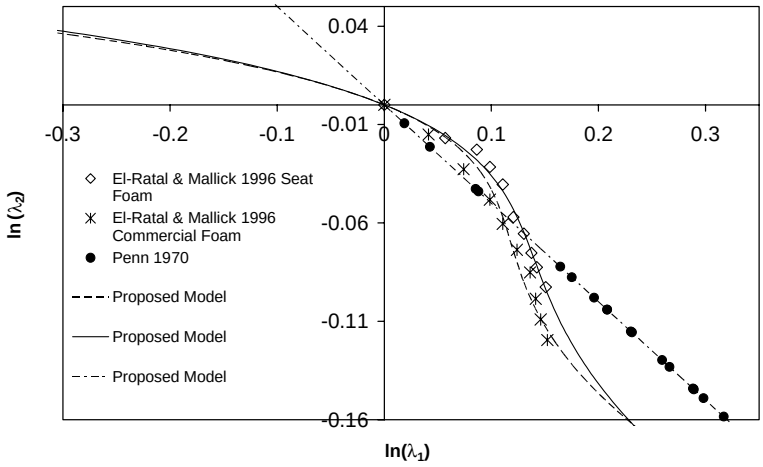

Fig. 18. Logarithmic Plot of Lateral Stretch versus Longitudinal Stretch.

A logarithmic plot of the longitudinal and lateral stretches would then show a linear relationship with a constant slope and pass through the origin. The results of Penn (1970) for the natural gum rubber show this trend as can be seen in Fig. 18. The apparent Poisson's ratio based on Eq. (51) for the two foams is also plotted in Fig. 18. The apparent Poisson's ratio for the two foams do not show a linear relationship but are highly nonlinear. The comparison with the proposed model for the apparent Poisson's ratio, however shows good agreement. The proposed model does fit the experimental results well for the range of tensile stretches tested but shows trends outside this range which can only be verified by further testing. Compression tests coupled with uniaxial tension on the same foam materials would help verify the proposed model and the material parameters chosen. Jemiolo and Turteltaub (2000) did provided extrapolated compression results for the foams tested by El-Ratal and Mallick (1996) but these have not been used here as they were not direct experimental results.

9. Conclusions

The strain energy density function for isotropic higher order elasticity developed in Attard (2003) has been extended here by deriving the higher order constitutive relationships for the second Piola Kirchhoff and Eulerian stress tensors as well as for their physical counterparts. General constitutive relationships have also been derived for the principal Lagrangian and Eulerian stresses in terms of the principal stretches. The strain energy density function is decomposed into a compressibility component being a generalization of the Simo and Pister (1984) proposal for neo-Hookean elasticity, and an incompressibility component being the generalized Mooney expression. Fundamental to the present proposal is the postulate that interaction or coupling between principal stresses only comes about through the compressibility component of the strain energy density function. In essence, the incompressibility component satisfies the Valanis–Landel hypothesis that the strain energy density for incompressible materials should be capable of representation as the sum of three separate but identical functions of each of the individual principal stretches. Vangerko and Treloar (1978) were able to test this hypothesis experimentally by applying large biaxial stretches on two batches of rubber. The predictions based on the proposed strain energy density compared very well with Vangerko's and Treloar's test results. Discrepancies could be explained by noting that stress-relaxation was present at the largest maintained stretches. Several experimental results for incompressible rubber-like

materials under homogeneous strain were also compared to the predictions of the proposed model and showed very good agreement.

Predictions for compressible materials under high hydrostatic compression were also good. The uniaxial tension experimental results of Penn (1970) involved measurements of stress versus stretches and volume changes in natural gum rubber. The proposed model compared well with the test results. The final comparison involved the tests conducted by El-Ratal and Mallick (1996) on two types of polyurethane foam under large deformation uniaxial tension. The volume change in the foams initially increased under tension and then showed a steady decrease with increasing tension. The proposed hyperelastic model could predict this response, as well as matching the stress versus stretch relationships. The hyperelastic model however predicted that eventually the volume change must increase with increasing tension so that as the longitudinal stretch approaches infinity($\lambda_1 \to +\infty$) the lateral stretch asymptotes towards zero ($\lambda_2 \to +0$) and the volume ratio approaches positive infinity ($J \to +\infty$). Further verification of the compressibility component of the proposed strain energy density function would be helped with the availability of compression and tension tests on the same foam material with measurements of volume changes at large longitudinal stretch.

Appendix A

Here we derive the expressions for the stresses for the common types of experiments performed on incompressible materials such as rubber under homogeneous strain with $\varsigma_{\mathrm{p}}^{33} = 0$ and $\lambda_3 = \frac{1}{\lambda_1 \lambda_2}$.

A.1. Uniaxial tension-compression

An incompressible material is stretched in the 1 direction with the lateral stresses $\varsigma_{\mathrm{p}}^{22}$ and $\varsigma_{\mathrm{p}}^{33}$ both zero. From Eq. (34) we can conclude that $\lambda_2 = 1/\sqrt{\lambda_1}$ and therefore the longitudinal stress derived using Eq. (33) is:

$$\varsigma_{\mathrm{p}}^{11} = s_{\mathrm{p}}^{11}\lambda_1 = \sum_{n=1}^{r} A_n(\lambda_1^{2n} - \lambda_1^{-n}) - B_n(\lambda_1^{-2n} - \lambda_1^{n}) \tag{A.1}$$

A.2. Equi-biaxial tension-compression

An incompressible material is stretched in two orthogonal directions 1 and 2. Hence $\lambda_1 = \lambda_2$ and $\varsigma_{\mathrm{p}}^{11} = \varsigma_{\mathrm{p}}^{22}$. Substituting into Eqs. (33) and (34) gives

$$\varsigma_{\mathrm{p}}^{11} = s_{\mathrm{p}}^{11}\lambda_1 = \sum_{n=1}^{r} A_n(\lambda_1^{2n} - \lambda_1^{-4n}) - B_n(\lambda_1^{-2n} - \lambda_1^{4n}) \tag{A.2}$$

A.3. Pure shear

This stress state can be modeled by subjecting a material to biaxial tension but with one of the principal stretches constrained to a fixed value of unity. Hence substituting $\lambda_2 = 1$ into Eqs. (33) and (34) yields:

$$\varsigma_{\mathrm{p}}^{11} = s_{\mathrm{p}}^{11}\lambda_1 = \sum_{n=1}^{r} (A_n + B_n)(\lambda_1^{2n} - (\lambda_1)^{-2n}) \tag{A.3}$$

$$\varsigma_{\mathrm{p}}^{22} = s_{\mathrm{p}}^{22} = \sum_{n=1}^{r} A_n(1 - (\lambda_1)^{-2n}) - B_n(1 - (\lambda_1)^{2n}) \tag{A.4}$$

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